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VORTICES AND INTERNAL WAVES IN A STRATIFIED FLUID†

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Equations describing the evolution of potential vortices and internal waves in a stably stratified fluid which fills a half-space are derived in Euler–Lagrange variables. Asymptotic series in a small parameter are constructed which give approximate solutions of the non-linear problem. The equations of the linear approximation for a potential vortex and internal waves are independent, while the equations of the higher-order approximations describe the interaction between the potential vortices and the internal waves. It is shown that distributed sources (which can be interpreted as intense atmospheric rainfall) cause an exponential increase in the potential vorticity, which in turn may lead to a considerable increase in the amplitudes of the internal waves. Asymptotic forms for the field of the internal waves in different space–time regions are constructed for the case of an exponential stratification.

1. THE EQUATIONS OF MOTION IN EULER–LAGRANGE VARIABLES

We consider the motion of an inviscid stably stratified fluid filling a half-space in the gravity field. The z axis of a Cartesian system of coordinates x, y, z is directed vertically upwards. In a state of equilibrium the density ρ_0 and the pressure p_0 are related by the equation $p_0'(z) = -g\rho_0(z)$. For an ideal gas the function $\alpha(z) = p_0(z)/\rho_0^\kappa(z)$ specifies the entropy distribution, where $\kappa = c_p/c_\upsilon > 1$. The condition of stability is $\rho'_0(z) \leqslant 0$ for an incompressible fluid and $\alpha'(z) \geqslant 0$ for an ideal gas. Without loss of generality we can assume that under stability conditions the strict inequalities hold, since we can always add strictly monotonic functions, which are as small as desired, to the functions $\rho_0(z)$ and $\alpha(z)$. It follows from the fact that the functions $\rho_0(z)$ and $\alpha(z)$ are strictly monotonic that the density (respectively, the entropy) of a fluid particle uniquely defines its distance from the coordinate plane xy in the equilibrium position.

Consider a fluid particle which, at the instant of time t, is situated at a point in space with coordinates x, y, z. In the equilibrium position the same particle will be situated a distance $\zeta(x, y, z, t)$ from the x, y coordinate plane. The function ζ retains its value in the particle and hence is the integral of the equations of motion. Assuming that this function is continuous and that $\partial\zeta/\partial z \geqslant c_0 > 0$, we will take x, y, ζ, t as the independent variables and take z as one of the dependent variables. The quantity $z - \zeta$ specifies the deviation of the fluid particle from the equilibrium position along the vertical. The density $\rho = \rho_0(\zeta)$ remains the same in the particle if the fluid is incompressible, or the entropy $\alpha = p/\rho^\kappa = \alpha(\zeta)$ if the fluid is an ideal gas.

Suppose (u, υ) are the horizontal components of the velocity vector. Using the notation

$$D = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \upsilon\frac{\partial}{\partial y}, \quad \Omega = \frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial y} \tag{1.1}$$

and employing simple considerations [1], we can convert Euler's equations and the continuity equation to the form

$$\frac{\partial u}{\partial t} - \upsilon\Omega + z_x D^2 z = -\frac{\partial H}{\partial x}, \quad \frac{\partial \upsilon}{\partial t} + u\Omega + z_y D^2 z = -\frac{\partial H}{\partial y} \tag{1.2}$$

$$z_\zeta D^2 z = -\frac{\partial H}{\partial \zeta} + n^2(\zeta)\left(H - \frac{1}{2}u^2 - \frac{1}{2}\upsilon^2\right) - N^2(\zeta) z \tag{1.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{D z_\zeta}{z_\zeta} + \frac{D\rho}{\rho} = \varepsilon Q(x, y, \zeta, t) \tag{1.4}$$

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where the function $\varepsilon Q(x, y, \zeta, t)$ denotes the distribution of the sources and sinks, ε is a small parameter and the functions $n^2(\zeta)$, H and $N^2(\zeta)$ are defined by different formulae for an incompressible fluid and for an ideal gas: for an incompressible fluid

$$H=\frac{p}{\rho}+gz+\frac{1}{2}(u^2+\upsilon^2),\quad n^2(\zeta)=-\frac{\rho_0'(\zeta)}{\rho_0(\zeta)},\quad N^2(\zeta)=-g\frac{\rho_0'(\zeta)}{\rho_0(\zeta)}$$

while for an ideal gas

$$H=\frac{\kappa}{\kappa-1}\,\frac{p}{p}+gz+\frac{1}{2}(u^2+\upsilon^2),\quad n^2(\zeta)=\frac{\alpha'(\zeta)}{\kappa\alpha(\zeta)},\quad N^2(\zeta)=\frac{h\alpha'(\zeta)}{\kappa\alpha(\zeta)}.$$

If the fluid is incompressible we have $D\rho = 0$ and the term $D\rho/\rho$ will not occur in Eq. (1.4). If the fluid is compressible and a is the velocity of sound, then for slow motions the quantity $D\rho/\rho = 2Da/a(\kappa-1)$ is small compared with $\delta\upsilon/\delta x$ and Du/Dy, and $D\rho/\rho$ can be omitted in Eq. (1.4). Hence, the difference between the case of a compressible and an incompressible fluid in this approximation will be the different definitions of the function H. In the Boussinesq approximation the quantity $n^2(\zeta)$ is assumed to be small and terms containing $n^2(\zeta)$ as a factor will be dropped from the equations. Henceforth we will confine ourselves to the Boussinesq approximation, although the case $n \neq 0$ does not differ from the case $n = 0$ except in unessential technical complications.

We will write expressions (1.1) in the form

$$D=\frac{\partial}{\partial t}+\frac{\partial\varphi}{\partial x}\,\frac{\partial}{\partial x}+\frac{\partial\varphi}{\partial y}\,\frac{\partial}{\partial y}+\frac{\partial(\psi,\cdot)}{\partial(x,y)},\quad \Omega=\Delta_2\psi \tag{1.5}$$

$$\frac{\partial u}{\partial x}+\frac{\partial\upsilon}{\partial y}=\Delta_2\varphi,\quad \Delta_2\varphi+\frac{Dz_\zeta}{z_\zeta}=\varepsilon Q$$

$$\left(u=\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y},\quad \upsilon=\frac{\partial\varphi}{\partial y}-\frac{\partial\psi}{\partial x},\quad \Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

Eliminating H from Eqs (1.2) we obtain

$$D\Omega+\Omega\left(\frac{\partial u}{\partial x}+\frac{\partial\upsilon}{\partial y}\right)+\frac{\partial(D^2z,z)}{\partial(x,y)}=0 \tag{1.6}$$

Using the identity

$$D\left(\frac{\partial(Dz,z)}{\partial(x,y)}\right)=\frac{\partial(D^2z,z)}{\partial(x,y)}-\left(\frac{\partial u}{\partial x}+\frac{\partial\upsilon}{\partial y}\right)\,\frac{\partial(Dz,z)}{\partial(x,y)}$$

and relations (1.5), we can write Eq. (1.6) in the form

$$D\Gamma=-\varepsilon Q\Gamma,\quad \Gamma=\frac{1}{z_\zeta}\left(\Omega+\frac{\partial(Dz,z)}{\partial(x,y)}\right) \tag{1.7}$$

It can be shown that in the independent variables x, y, z, t

$$\Gamma=(\operatorname{rot}\mathbf{v},\nabla\zeta) \tag{1.8}$$

Formula (1.7) can be regarded as an extension of Ertel's formula [3] to the case when sources and sinks are distributed in space. In the region of space where there are no sources and sinks the quantity (rot v, $\nabla\rho$) keeps its value in liquid particles; it is usually called a potential vortex [4]. Since $\nabla\rho = \rho'(\zeta)\nabla\zeta$, the quantity Γ henceforth also be called a potential vortex. Note that if we dispense with the hypothesis of the slow motion of an ideal gas and we do not drop the term $D\rho/\rho$ in the continuity equation, Eq. (1.7) remains true but we must put the following in it

$$\Gamma = \frac{1}{\rho z_\zeta}\left(\Omega + \frac{\partial(Dz, z)}{\partial(x, y)}\right)$$

Applying the operator Δ_2 to Eq. (1.3) and finding $\Delta_2 H$ from Eqs (1.2), we obtain the equation

$$\frac{\partial^2}{\partial\zeta\partial t}\left(\frac{Dz_\zeta}{z\zeta}\right) + \Delta_2(z_\zeta D^2 z + N^2(\zeta)z) =$$

$$= \varepsilon\frac{\partial^2 Q}{\partial\zeta\partial t} + \frac{\partial}{\partial\zeta}\left(\frac{\partial(\varphi,\Omega)}{\partial(x,y)} + \Delta_2 z D^2 z + (\nabla_2\psi, \nabla_2\Omega) + (\nabla_2 z, \nabla_2 D^2 z)\right) \tag{1.9}$$

The system of three non-linear equations (1.7), (1.9) and (1.5) is obtained for the three unknown functions z, φ, ψ.

We will take the initial conditions in the form

$$z|_{t=0} = \zeta + \varepsilon\omega_0(x, y, \zeta), \quad Dz|_{t=0} = \varepsilon\omega_1(x, y, \zeta), \quad \Gamma|_{t=0} = \varepsilon\gamma_0(x, y, \zeta) \tag{1.10}$$

Hence, the deviation of the particle along the vertical from the equilibrium position, the vertical velocity of the particle and the potential vortex at the initial instant are given. The functions ω_0, ω_1, γ_0 and Q are continuous and finite (or fairly rapidly decreasing at infinity). It follows from (1.5) and (1.7) that to determine the initial values of the functions φ and ψ we must solve a plane Poisson equation with finite and continuous right-hand side, or with a right-hand side which approaches zero fairly rapidly at infinity.

Suppose (r, θ) are polar coordinates in the plane. In the axisymmetric case the functions φ, ψ and z depend only on r, ζ and t, and the equations and boundary conditions take the form

$$\Delta_2\varphi + \frac{1}{z_\zeta}\left(\frac{\partial^2 z}{\partial t\partial\zeta} + \frac{\partial\varphi}{\partial r}\frac{\partial^2 z}{\partial\zeta\partial r}\right) = \varepsilon Q(r, \zeta, t)$$

$$\frac{\partial\Gamma}{\partial t} + \frac{\partial\varphi}{\partial r}\frac{\partial\Gamma}{\partial r} = -\varepsilon\Gamma Q(r, \zeta, t)$$

$$\frac{\partial^2}{\partial\zeta\partial t}\left(\frac{1}{z_\zeta}\left(\frac{\partial^2 z}{\partial\zeta\partial t} + \frac{\partial\varphi}{\partial r}\frac{\partial^2 z}{\partial\zeta\partial r}\right)\right) + \Delta_2(z_\zeta D^2 z + N^2(\zeta)z) =$$

$$= \varepsilon\frac{\partial^2 Q}{\partial\zeta\partial t} + \frac{\partial}{\partial\zeta}\left(\frac{\partial\psi}{\partial r}\frac{\partial(\Gamma z_\zeta)}{\partial r} - (\Gamma z_\zeta)^2 + \Delta_2 z D^2 z + \frac{\partial z}{\partial r}\frac{\partial}{\partial r}(D^2 z)\right) \tag{1.11}$$

$$\Delta_2\psi = -\Gamma z_\zeta, \quad \Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

$$D^2 = \frac{\partial^2}{\partial t} + 2\frac{\partial\varphi}{\partial r}\frac{\partial^2}{\partial t\partial r} + \left(\frac{\partial^2\varphi}{\partial t\partial r} + \frac{\partial\varphi}{\partial r}\frac{\partial^2\varphi}{\partial r^2}\right)\frac{\partial}{\partial r} + \left(\frac{\partial\varphi}{\partial r}\right)^2\frac{\partial^2}{\partial r^2}$$

If the fluid is bounded below by a fixed solid wall, we will take the boundary condition in the form

$$z|_{\zeta=0} = \varepsilon\omega_2(x, y, t) \tag{1.12}$$

It follows from condition (1.12) that particles of the same density are situated on the solid wall.

2. CONSTRUCTION OF THE SOLUTION IN THE FORM OF SERIES IN SMALL PARAMETERS

Suppose the region $G \subset R^3$ is bounded and has axial symmetry, and suppose the function Q has a carrier in $G[0, T]$. The development of a potential vortex is related not only to the value of the parameter ε, characterizing the power of the sources, but also to the amount of fluid produced at a given point of space in the time t for which the source acts. Taking this into account we introduce the functional parameter

$$\tau = \varepsilon \exp\left(-\varepsilon \int_0^t Q(r,\zeta,t')dt'\right) \tag{2.1}$$

and we will seek a formal solution of Eqs (1.11) with initial conditions (1.10) and boundary condition (1.12) in the form of series in powers of the parameters τ and ε

$$\begin{aligned} &z = \zeta + \varepsilon z_0 + \sum_{k=2}^{\infty} z_k \tau^k, \quad z_k = \sum_{m=0}^{\infty} z_{km}\varepsilon^m \\ &\varphi = \varepsilon\varphi_0 + \sum_{k=2}^{\infty} \varphi_k \tau^k, \quad \varphi_k = \sum_{m=0}^{\infty} \varphi_{km}\varepsilon^m \\ &\Gamma = \sum_{k=1}^{\infty} \Gamma_k \tau^k, \quad \Gamma_k = \sum_{m=0}^{\infty} \Gamma_{km}\varepsilon^m, \quad \psi = \sum_{k=1}^{\infty} \psi_k \tau^k, \quad \psi_k = \sum_{m=0}^{\infty} \psi_{km}\varepsilon^m \end{aligned} \tag{2.2}$$

Substituting series (2.2) into Eq. (1.11), and the initial and boundary conditions (1.10) and (1.12), we obtain a sequence of boundary-value problems for determining the unknown functions z_{km}, φ_{km}, Γ_{km}, ψ_{km}.

To determine the function z_{00} we must solve a mixed problem

$$\begin{aligned} &\mathrm{L}\, z_{00} = \frac{\partial^2 Q}{\partial\zeta\partial t}, \quad L = \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + N^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \\ &z_{00}|_{t=0} = \omega_0(r,\zeta), \quad \left.\frac{\partial z_{00}}{\partial t}\right|_{t=0} = \omega_1(r,\zeta), \quad z_{00}|_{\zeta=0} = \omega_2(r,\zeta) \end{aligned} \tag{2.3}$$

(L is the differential operator of the internal waves).

For the limitations imposed on the known functions, the existence and uniqueness of the solution of problem (2.3) with the additional condition that the velocity field is bounded at infinity, can be established by methods of mathematical physics. For the case when N = const the solution is expressed explicitly in terms of convolutions with the fundamental solution of the equation $\mathrm{L}u = 0$ [4].

We will write the solution of problem (2.3) in the form

$$z_{00} = \mathrm{A}Q + \mathrm{B}\omega_0 + \mathrm{C}\omega_1 + \mathrm{D}\omega_2 \tag{2.4}$$

Explicit expressions for the linear operators A, B, C and D will be derived in the next section.

To determine the function φ_{00} we must obtain the bounded solution of Poisson's equation in the plane

$$\Delta_2\varphi_{00} = Q(r,\zeta,t) - \partial^2 z_{00} / \partial t\partial\zeta$$

We know that for a continuous right-hand side, decreasing rapidly at infinity, a solution of Poisson's equation exists and is unique.

To determine Γ_{10} and ψ_{10} we obtain the equation

$$\partial\Gamma_{10} / \partial t = 0, \quad \Gamma_{10}|_{t=0} = \gamma_0(r,\zeta), \quad \Delta_2\psi_{10} = -\gamma_0(r,\zeta)$$

from which it is clear that $\Gamma_{10} = \gamma_0(r, \zeta)$ and $\psi_{10} = -\Delta_2^{-1}\gamma_0$.

To determine z_{20} we obtain the mixed problem

$$\begin{aligned} &\mathrm{L}\, z_{20} = \frac{\partial}{\partial\zeta}\left(\frac{\partial\psi_{10}}{\partial r}\frac{\partial\gamma_0}{\partial r} - \gamma_0^2\right) \\ &z_{20}|_{t=0} = 0, \quad \left.\frac{\partial z_{20}}{\partial t}\right|_{t=0} = 0, \quad z_{20}|_{\zeta=0} = 0 \end{aligned}$$

The function φ_{20} is the solution of Poisson's equation

$$\Delta_2\varphi_{20} = -\partial z_{20} / \partial t \partial \zeta$$

Taking (2.4) into account we obtain

$$z_{20} = A\left(\frac{\partial}{\partial \zeta}\left(\frac{\partial \psi_{10}}{\partial r}\frac{\partial \gamma_0}{\partial r} - \gamma_0^2\right)\right), \quad \varphi_{20} = -\Delta_0^{-1}\left(\frac{\partial^2 z_{20}}{\partial t \partial \zeta}\right) \tag{2.5}$$

The determination of the remaining terms of series (2.2) reduces to solving inhomogeneous boundary-value problems for the operators L, Δ_2 and $\partial/\partial t$ with right-hand sides which depend on the previously obtained terms of these series, so that series (2.2) can be constructed, in principle. The principal terms of the asymptotic form with respect to the parameters ε and τ have the form

$$z = \zeta + \varepsilon z_{00} + \tau^2 z_{20}, \quad \varphi = \varepsilon\varphi_{00} + \tau^2\varphi_{20}, \quad \Gamma = \gamma_0(r,\zeta)\tau, \quad \psi = -\tau\Delta_2^{-1}\gamma_0 \tag{2.6}$$

where the parameter τ is given by (2.1) while z_{00} and z_{20} are given by Eqs (2.4) and (2.5).

If the function $Q(r, \zeta, t)$ is negative (i.e. the sinks are distributed over space), the functional parameter τ defined by the function (2.1) will increase exponentially with time and the value of τ^2 may become comparable with or exceed the value of ε. Hence, the principal term of the asymptotic form must be taken in the form (2.6), and it is insufficient to confine ourselves solely to the term $\zeta + \varepsilon z_{00}$, which is found from the linear theory of internal waves. The linear theory will correctly describe the situation only when there are no potential vortices or they are fairly small in the specified region of space.

Note again that in meteorology sinks can sometimes be interpreted as intense atmospheric rainfall. Evaporation from the ocean surface and convection lead to the formation of vast cloud areas above the ocean surface. Complex and so-far insufficiently investigated processes occur in a cloud, including evaporation and condensation of water vapour. The condensation process is accompanied, under certain conditions by precipitation in the form of rain, snow and hail. To a first approximation the precipitation of intense rain can be interpreted as the occurrence of sinks within the clouds. According to the theory described above, the potential vortices present in the atmosphere may, on being exponentially amplified, gather strength and lead to the formation of powerful tornado-type vortices. Other mechanisms are also known for the formation of atmospheric vortices related to wind shifts and the turning of an unstable vortex sheet into a ring vortex. Which process plays a fundamental role or which combination of processes leads to the actual occurrence of the vortex in the atmosphere? Further investigations are needed to give an answer to these questions.

3. INVESTIGATION OF THE OPERATORS WHICH GIVE A SOLUTION OF THE LINEAR PROBLEM FOR N = CONST

It follows from the results in Section 2 that the determination of the function z_{mk} reduces to solving the following mixed problem

$$\mathrm{L}u = f_3(x,y,\zeta,t) \tag{3.1}$$

$$u|_{t=0} = f_0(x,y,\zeta), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = f_1(x,y,\zeta), \quad u|_{\zeta=0} = f_2(x,y,t)$$

where the functions f_i are continuous and finite (or fairly rapidly decreasing at infinity). Similar problems have been investigated by many researchers. The solution is usually expressed in terms of different convolutions of the functions f_i with the fundamental solution of the equation $\mathrm{L}u = 0$. A fairly complete investigation of the fundamental solution and a detailed bibliography are given in [5, 6]. The simplest way of solving the boundary-value problem (3.1), unlike that used in [5], is based on the fact that after using a Laplace transformation in time and a simple change of variable the problem reduces to solving a Dirichlet problem for Poisson's equation in a half-space, which can be expressed in a known way in terms of potentials. By carrying out an inverse Laplace transformation we obtain a solution of problem (3.1) in the form of convolutions of the functions f_i with the fundamental solution of the equation $\mathrm{L}u = 0$.

The fundamental solution for the half-space has the form

$$F(x,y,\zeta,\zeta',t)=\Phi(x,y,\zeta-\zeta',t)-\Phi(x,y,\zeta+\zeta',t) \tag{3.2}$$

$$\Phi(x,y,\zeta,t)=-\frac{1}{4\pi N}\frac{1}{\sqrt{x^2+y^2+\zeta^2}}\omega\left(\frac{\zeta}{\sqrt{x^2+y^2+\zeta^2}},\ Nt\right),\quad k^2+\lambda^2=1$$

$$\omega(\lambda,\tau)=\frac{2}{\pi}\int_{|\lambda|}^{1}\frac{\sin(\tau\xi)d\xi}{\sqrt{(1-\xi)^2(\xi^2-\lambda^2)}}=\frac{2}{\pi}\int_0^{\pi/2}\frac{\sin(\tau\sqrt{1-k^2\cos^2 u})}{\sqrt{1-k^2\cos^2 u}}du$$

The solution of the mixed problem (3.1) can be written in the form of convolutions with the fundamental solution

$$u=\sum_{k=0}^{4}\mathrm{A}_k f_k \tag{3.3}$$

$$\mathrm{A}_0 f_0=\iiint_{R^3}\frac{\partial F}{\partial t}(x-x',y-y',\zeta,\zeta',t)\Delta f_0(x',y',\zeta')dx'dy'd\zeta'$$

$$\mathrm{A}_1 f_1=\iiint_{R^3}F(x-x',y-y',\zeta,\zeta',t)\Delta f_1(x',y',\zeta')dx'dy'd\zeta'$$

$$\mathrm{A}_2 f_2=-\int_0^t dt'\iint_{R^2}\frac{\partial}{\partial\zeta}\Phi(x-x',y-y',\zeta,t-t')f_2(x',y',t')dx'dy'$$

$$\mathrm{A}_3 f_3=\int_0^t dt'\iiint_{R^3}F(x-x',y-y',\zeta,\zeta't-t')f_3(x',y',\zeta',t')dx'dy'd\zeta'$$

It follows from (3.2) that the function $\omega(\lambda,\tau)$, for which various researchers have derived a variety of accurate and approximate solutions, plays the main role when constructing the fundamental solution. We will give without proof some seemingly new formulae for the function $\omega(\lambda,\tau)$

$$\omega(\lambda,\tau)=2\sum_{n=0}^{\infty}J_{2n+1}(\tau)P_n(1-2\lambda^2)=\int_0^{\tau}J_0(u)du+2\sum_{n=0}^{\infty}(P_n(1-2\lambda^2)-1)J_{2n+1}(\tau)=$$
$$=\sin\tau+2\sum_{n=0}^{\infty}(P_n(1-2\lambda^2)-(-1)^n)J_{2n+1}(\tau) \tag{3.4}$$

where $J_n(\tau)$ is a Bessel function and $P_n(z)$ is a Legendre polynomial. For fixed τ formulae (3.4) define the asymptotic form of the function $\omega(\lambda,\tau)$ as $\lambda\to 0$ and $\lambda\to 1$.

When $0<\lambda_0\leqslant\lambda\leqslant 1$ the following asymptotic formula holds as $\tau\to+\infty$ uniformly with respect to λ

$$\omega(\lambda,\tau)=2\sin\kappa^+(A_0(\lambda)J_0(\kappa^-)+\frac{1}{\kappa^-}A_1(\lambda)J_1(\kappa^-))-$$
$$-2(1-\lambda)\cos\kappa^+(B_0(\lambda)J_1(\kappa^-)+\frac{1}{\kappa^-}B_1(\lambda)J_2(\kappa^-))$$

$$\kappa^{\pm}=\frac{1\pm\lambda}{2}\tau,\quad A_0=\frac{1+\sqrt{\lambda}}{\sqrt{8\lambda(1+\lambda)}},\quad B_0=\frac{1}{(1+\sqrt{\lambda})\sqrt{8\lambda(1+\lambda)}}$$

$$A_1=-\frac{1}{2}\frac{1+\sqrt{\lambda}+4\lambda+\lambda^{3/2}+\lambda^2}{(1+\sqrt{\lambda})(8\lambda(1+\lambda))^{3/2}},\quad B_1=-\frac{1+3\sqrt{\lambda}+3\lambda^{3/2}+\lambda^2}{2(1+\sqrt{\lambda})^3(8\lambda(1+\lambda))^{3/2}}$$

When $0<\delta\leqslant\lambda\leqslant 1-\delta$ and $\tau\to+\infty$ the following simpler asymptotic formula holds

$$\sqrt{\frac{\pi}{2}}\omega(\lambda,\tau)=\frac{1}{\sqrt{\tau(1-\lambda^2)}}\sin\left(\tau-\frac{\pi}{4}\right)+\frac{1}{\sqrt{\tau\lambda(1-\lambda^2)}}\sin\left(\lambda\tau+\frac{\pi}{4}\right)-$$

$$-\frac{5-\lambda^2}{8(1-\lambda^2)^{3/2}\tau^{3/2}}\sin\left(\tau+\frac{\pi}{4}\right)-\frac{5\lambda^2-1}{8(\lambda(1-\lambda^2))^{3/2}\tau^{3/2}}\sin\left(\lambda\tau-\frac{\pi}{4}\right)-$$
$$-\frac{3(43+2\lambda^2+3\lambda^4)}{128(1-\lambda^2)^{5/2}\tau^{5/2}}\sin\left(\tau-\frac{\pi}{4}\right)-\frac{3(43\lambda^4+2\lambda^2+3)}{128(\lambda(1-\lambda^2))^{5/2}\tau^{5/2}}\sin\left(\lambda\tau+\frac{\pi}{4}\right)+\ldots \tag{3.5}$$

It follows from (3.2) and (3.5) that in a cone lying inside the first octant with vertex at the origin of coordinates as $Nt\to\infty$ the following asymptotic formula holds ($\rho^2 = x^2 + y^2$, $R^2 = \rho^2 + z^2$)

$$-(2\pi)^{3/2}N\Phi(x,y,z,t)=-\frac{1}{\rho\sqrt{Nt}}\sin\left(Nt-\frac{\pi}{4}\right)+\frac{\sqrt{R}}{\rho\sqrt{Ntz}}\sin\left(\frac{Ntz}{R}+\frac{\pi}{4}\right)-$$
$$-\frac{5\rho^2+4z^2}{8(\rho\sqrt{Nt})^3}\sin\left(Nt+\frac{\pi}{4}\right)-\frac{(4z^2-\rho^2)}{8(\rho\sqrt{Ntz})^3}\sin\left(\frac{Ntz}{R}-\frac{\pi}{4}\right)-$$
$$-\frac{3(43\rho^4+88z^2\rho^2+48z^4)}{128(\rho\sqrt{Nt})^5}\sin\left(Nt-\frac{\pi}{4}\right)-\frac{3(48z^4+8z^2\rho^2+3\rho^4)R^{5/2}}{128(\rho\sqrt{Ntz})^5}\sin\left(\frac{Ntz}{R}+\frac{\pi}{4}\right) \tag{3.6}$$

In (3.3) the fundamental solution is convoluted with infinitely differentiable and finite functions. Suppose their carrier is situated in a region Ω; then in the set

$$G=\left\{(x,y,z):\ 0<\lambda_0\leq\min_{\Omega}\lambda_\pm\leq\max_{\Omega}\lambda_\pm\leq 1-\lambda_0\right\}$$
$$\left(\lambda_\pm=\frac{z\pm z'}{\sqrt{(x-x')^2+(y-y')^2+(z\pm z')^2}}\right)$$

the asymptotic form of the solution of problem (3.1) as $Nt\to\infty$ is obtained by substituting the asymptotic representation (3.6) into (3.2). After integrating over the region Ω, terms containing harmonics of $\sin(Nt\lambda_\pm)$ begin to decrease more rapidly than any negative power of Nt, since the phase of these harmonics has no stationary points while the functions f_i are finite and infinitely differentiable. Assuming $f_2 = f_3 = 0$ we obtain the asymptotic approximation for the solution of the Cauchy problem in the form

$$u(x,y,z,t)=-\left(\frac{2}{\pi Nt}\right)^{3/2}z\left(\Gamma_1(x,y),\sin\left(Nt+\frac{\pi}{4}\right)+N\Gamma_0(x,y)\cos\left(Nt+\frac{\pi}{4}\right)\right)$$
$$\Gamma_i(x,y)=\iiint_{\Omega}\frac{z'f_i(x',y',z')dx'dy'dz'}{((x-x')^2+(y-y')^2)^{3/2}}$$

One can use (3.4) to obtain approximate formulae for $(x, y, z) \notin G$.

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